Chapter 6

Eigenvalues and Eigenvectors

Eigenvalues and **eigenvectors** have new information about a square matrix—deeper than its rank or its column space. We look for **eigenvectors** $\boldsymbol{x}$ that don't change direction when they are multiplied by A. Then $\boldsymbol{Ax = \lambda x}$ with **eigenvalue** $\boldsymbol{\lambda}$. (You could call λ the stretching factor.) Multiplying again gives $\boldsymbol{A^2 x = \lambda^2 x}$. We can go onwards to $\boldsymbol{A^{100} x = \lambda^{100} x}$. And we can combine two or more eigenvectors:

$$A(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2 \qquad A^2(c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2) = c_1\lambda_1^2\boldsymbol{x}_1 + c_2\lambda_2^2\boldsymbol{x}_2$$

When we separate the input into eigenvectors, each eigenvector just goes its own way.

The eigenvalues are the growth factors in $\boldsymbol{A^n x = \lambda^n x}$. If all $|\lambda_i| < 1$ then A^n will eventually approach zero. If any $|\lambda_i| > 1$ then A^n eventually grows. If $\lambda = 1$ then $A^n\boldsymbol{x}$ never changes (a steady state). For the economy of a country or a company or a family, the size of λ is a critical number.

Properties of a matrix are reflected in the properties of the λ's and the $\boldsymbol{x}$'s. A symmetric matrix S has perpendicular eigenvectors—and all its eigenvalues are real numbers. The kings of linear algebra are **symmetric matrices with positive eigenvalues**. These "positive definite matrices" signal a minimum point for a function like the energy $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}S\boldsymbol{x}$. That is the n-dimensional form of the calculus test $d^2f/dx^2 > 0$ for a minimum of $f(x)$.

This chapter ends by solving linear differential equations $d\boldsymbol{u}/dt = A\boldsymbol{u}$. The pieces of the solution are $\boldsymbol{u}(t) = e^{\lambda t}\boldsymbol{x}$ instead of $\boldsymbol{u}_n = \lambda^n\boldsymbol{x}$—exponentials instead of powers. The whole solution is $\boldsymbol{u}(t) = e^{At}\boldsymbol{u}(0)$. For linear differential equations with a constant matrix A, please use its eigenvectors.

Section 6.4 gives the rules for complex matrices—including the famous **Fourier matrix**.

6.1 Introduction to Eigenvalues : $\boldsymbol{Ax} = \boldsymbol{\lambda x}$

1 If $\boxed{\boldsymbol{Ax} = \boldsymbol{\lambda x}}$ then $\boldsymbol{x} \neq \mathbf{0}$ is an **eigenvector** of A and the number λ is the **eigenvalue**.

2 Then $\boxed{\boldsymbol{A^n x} = \boldsymbol{\lambda^n x}}$ for every n and $(A + cI)\boldsymbol{x} = (\lambda + c)\boldsymbol{x}$ and $A^{-1}\boldsymbol{x} = \boldsymbol{x}/\lambda$ if $\lambda \neq 0$.

3 $(A - \lambda I)\boldsymbol{x} = \mathbf{0} \Rightarrow$ **the determinant of $\boldsymbol{A - \lambda I}$ is zero : this equation produces $\boldsymbol{n}$ $\boldsymbol{\lambda}$'s**.

4 Check λ's by $(\boldsymbol{\lambda_1})(\boldsymbol{\lambda_2}) \cdots (\boldsymbol{\lambda_n}) = \mathbf{det}\, \boldsymbol{A}$ and $\boldsymbol{\lambda_1} + \cdots + \boldsymbol{\lambda_n} =$ diagonal sum $a_{11} + \cdots + a_{nn}$.

5 Projections have $\lambda = 1$ and 0. Rotations have $\lambda = e^{i\theta}$ and $e^{-i\theta}$: *complex numbers* !

This chapter enters a new part of linear algebra. The first part was about $A\boldsymbol{x} = \boldsymbol{b}$: linear equations for a steady state. Now the second part is about **change**. Time enters the picture—continuous time in a differential equation $\boldsymbol{du/dt} = A\boldsymbol{u}$ or time steps $k = 1, 2, 3, \ldots$ in $\boldsymbol{u}_{k+1} = A\boldsymbol{u}_k$. Those equations are NOT solved by elimination.

We want "eigenvectors" $\boldsymbol{x}$ that don't change direction when you multiply by $\boldsymbol{A}$. The solution vector $\boldsymbol{u}(t)$ or $\boldsymbol{u}_k$ stays in the direction of that fixed vector $\boldsymbol{x}$. Then we only look for the number (changing with time) that multiplies $\boldsymbol{x}$: a one-dimensional problem.

A good model comes from the powers $A, A^2, A^3, \ldots$ of a matrix. Suppose you need the hundredth power A^{100}. Its columns are very close to the **eigenvector** $\boldsymbol{x} = (\mathbf{.6},\ \mathbf{.4})$:

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{.8} & \mathbf{.3} \\ \mathbf{.2} & \mathbf{.7} \end{bmatrix} \quad A_2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad A_3 = \begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix} \qquad \boldsymbol{A}^{\mathbf{100}} \approx \begin{bmatrix} \mathbf{.6000} & \mathbf{.6000} \\ \mathbf{.4000} & \mathbf{.4000} \end{bmatrix}$$

A^{100} was found by using the **eigenvalues $\mathbf{1}$ and $\mathbf{1/2}$** of this A, not by multiplying 100 matrices. Eigenvalues and eigenvectors are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors. Almost all vectors will change direction, when they are multiplied by A. ***Certain exceptional vectors $\boldsymbol{x}$ are in the same direction as $A\boldsymbol{x}$. Those are the "eigenvectors"***. Multiply an eigenvector by A, and the vector $A\boldsymbol{x}$ is a number λ times the original $\boldsymbol{x}$.

The basic equation is $A\boldsymbol{x} = \lambda\boldsymbol{x}$. The number λ is an eigenvalue of A.

The eigenvalue λ tells whether the special vector $\boldsymbol{x}$ is stretched or shrunk or reversed— when it is multiplied by A. We may find $\lambda = 2$ or $\frac{1}{2}$ or -1. The eigenvalue λ could be zero ! Then $A\boldsymbol{x} = 0\boldsymbol{x}$ means that this eigenvector $\boldsymbol{x}$ is in the nullspace of A.

If A is the identity matrix, every vector has $A\boldsymbol{x} = \boldsymbol{x}$. All vectors are eigenvectors of I. All eigenvalues "lambda" are $\lambda = 1$. This is unusual to say the least. Most 2 by 2 matrices have *two* eigenvector directions and *two* eigenvalues. We will show that $\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda I}) = \mathbf{0}$.

This section explains how to compute the $\boldsymbol{x}$'s and λ's. It can come early in the course. We only need the determinant $\boldsymbol{ad} - \boldsymbol{bc}$ of a 2 by 2 matrix. Example 1 uses $\det(A - \lambda I) = 0$ to find the eigenvalues $\lambda = 1$ and $\lambda = \frac{1}{2}$ for the matrix A that appears above.

Example 1 The key numbers in $\det(A - \lambda I)$ are $\mathbf{.8} + \mathbf{.7} = \mathbf{1.5} = \frac{\mathbf{3}}{\mathbf{2}}$ on the diagonal of A and $(\mathbf{.8})\,(\mathbf{.7}) - (\mathbf{.3})\,(\mathbf{.2}) = \mathbf{.5} =$ **determinant of** $\boldsymbol{A} = \frac{\mathbf{1}}{\mathbf{2}}$:

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{.8} & \mathbf{.3} \\ \mathbf{.2} & \mathbf{.7} \end{bmatrix} \quad \det \begin{bmatrix} \mathbf{.8} - \boldsymbol{\lambda} & \mathbf{.3} \\ \mathbf{.2} & \mathbf{.7} - \boldsymbol{\lambda} \end{bmatrix} = \lambda^2 - \frac{\mathbf{3}}{\mathbf{2}}\lambda + \frac{\mathbf{1}}{\mathbf{2}} = (\lambda - 1)\left(\lambda - \frac{1}{2}\right).$$

I factored the quadratic into $\lambda - 1$ times $\lambda - \frac{1}{2}$, to see the two eigenvalues $\boldsymbol{\lambda} = \mathbf{1}$ and $\frac{\mathbf{1}}{\mathbf{2}}$. **This tells us that $\boldsymbol{A} - \boldsymbol{I}$ and $\boldsymbol{A} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{I}$ (with zero determinant) are not invertible**. The eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are in the nullspaces of $A - I$ and $A - \frac{1}{2}I$.

$(A - I)\boldsymbol{x}_1 = 0$ is $A\boldsymbol{x}_1 = \boldsymbol{x}_1$. That first eigenvector is $\boldsymbol{x}_1 = (\mathbf{.6},\ \mathbf{.4})$.

$(A - \frac{1}{2}I)\boldsymbol{x}_2 = 0$ is $A\boldsymbol{x}_2 = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{x}_2$. That second eigenvector is $\boldsymbol{x}_2 = (\mathbf{1},\ \mathbf{-1})$:

$$\boldsymbol{x}_1 = \begin{bmatrix} \mathbf{.6} \\ \mathbf{.4} \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad (A\boldsymbol{x}_1 = \boldsymbol{x}_1 \text{ means that } \boldsymbol{\lambda_1 = 1})$$

$$\boldsymbol{x}_2 = \begin{bmatrix} \mathbf{1} \\ \mathbf{-1} \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \tfrac{1}{2}\,\boldsymbol{x}_2 \text{ so } \boldsymbol{\lambda_2} = \tfrac{\mathbf{1}}{\mathbf{2}}).$$

From $A\boldsymbol{x}_1 = \boldsymbol{x}_1$ we get $A^2\boldsymbol{x}_1 = A\boldsymbol{x}_1 = \boldsymbol{x}_1$. Every power of A will have $A^n\boldsymbol{x}_1 = \boldsymbol{x}_1$. Multiplying $\boldsymbol{x}_2$ by A gave $\frac{1}{2}\boldsymbol{x}_2$, and if we multiply again we get $A^2\boldsymbol{x}_2 = (\frac{1}{2})^2$ times $\boldsymbol{x}_2$.

The eigenvectors of A remain eigenvectors of $\boldsymbol{A^2}$. The eigenvalues $\boldsymbol{\lambda}$ are squared.

This pattern succeeds because the eigenvectors $\boldsymbol{x}_1, \boldsymbol{x}_2$ stay in their own directions. The eigenvalues of A^{100} are $1^{100} = 1$ and $(\frac{1}{2})^{100} =$ very small number.

An eigenvector $\boldsymbol{x}$ of $\boldsymbol{A}$ is also an eigenvector of every $\boldsymbol{A^n}$. Then $\boldsymbol{A^n x = \lambda^n x}$.

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The first column $(\mathbf{.8}, \mathbf{.2})$ of A is the combination $(\mathbf{.6}, \mathbf{.4}) + (\mathbf{.2}, \mathbf{-.2})$.

Separate column 1 of $\boldsymbol{A}$ into eigenvectors

$$\begin{bmatrix} \mathbf{.8} \\ \mathbf{.2} \end{bmatrix} = \begin{bmatrix} \mathbf{.6} \\ \mathbf{.4} \end{bmatrix} + \begin{bmatrix} \mathbf{.2} \\ \mathbf{-.2} \end{bmatrix} = \boldsymbol{x}_1 + \mathbf{.2}\,\boldsymbol{x}_2 \tag{1}$$

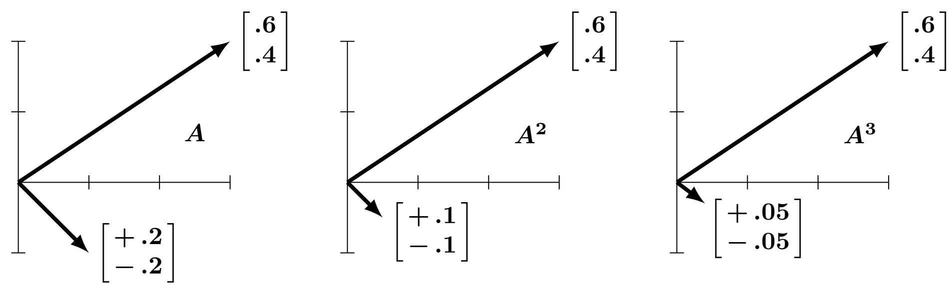


Figure 6.1: The first columns of A, A^2, A^3 are $\begin{bmatrix} \mathbf{.8} \\ \mathbf{.2} \end{bmatrix}, \begin{bmatrix} \mathbf{.7} \\ \mathbf{.3} \end{bmatrix}, \begin{bmatrix} \mathbf{.65} \\ \mathbf{.35} \end{bmatrix}$ approaching $\begin{bmatrix} \mathbf{.6} \\ \mathbf{.4} \end{bmatrix}$.

$$\textbf{Multiply each } \boldsymbol{x_i} \textbf{ by } \boldsymbol{\lambda_i} \textbf{ for column 1 of } \boldsymbol{A^2} \qquad \boldsymbol{A}\begin{bmatrix} \mathbf{.8} \\ \mathbf{.2} \end{bmatrix} \quad \text{is} \quad \boldsymbol{x}_1 + \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{.2})\boldsymbol{x}_2 = \begin{bmatrix} \mathbf{.6} \\ \mathbf{.4} \end{bmatrix} + \begin{bmatrix} \mathbf{.1} \\ -\mathbf{.1} \end{bmatrix} = \begin{bmatrix} \mathbf{.7} \\ \mathbf{.3} \end{bmatrix} \quad (2)$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A. At every step $\boldsymbol{x}_1$ is unchanged, because $\lambda_1 = 1$. But $\boldsymbol{x}_2$ is multiplied 99 times by $\lambda_2 = \frac{1}{2}$:

$$\textbf{Column 1 of } \boldsymbol{A^{100}} \qquad \boxed{A^{99}\begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \text{is really} \quad \boldsymbol{x}_1 + (\mathbf{.2})(\tfrac{\mathbf{1}}{\mathbf{2}})^{\mathbf{99}}\boldsymbol{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} \text{very} \\ \text{small} \\ \text{vector} \end{bmatrix}.} \quad (3)$$

This is the first column of A^{100}. The number we originally wrote as .6000 was not exact. We left out (.2) times $(\frac{1}{2})^{99}$ which wouldn't show up for 30 decimal places.

The eigenvector $\boldsymbol{x}_1$ is a "steady state" that doesn't change (because it has $\boldsymbol{\lambda_1 = 1}$). The eigenvector $\boldsymbol{x}_2$ is a "decaying mode" that virtually disappears (because $\boldsymbol{\lambda_2 = .5}$). The higher the power of A, the more closely its columns approach the steady state.

This particular A is called a ***Markov matrix***. Its largest eigenvalue is $\lambda = 1$. Its eigenvector $\boldsymbol{x}_1 = (.6, .4)$ is the *steady state*—which all columns of A^k will approach. *A giant Markov matrix is the key to Google's superfast web search.*

For projection matrices P, the column space projects to itself $(P\boldsymbol{x} = \boldsymbol{x})$. The nullspace of P projects to zero $(P\boldsymbol{x} = 0\boldsymbol{x})$. The eigenvalues of P are $\lambda = 1$ and $\lambda = 0$.

Example 2 **The projection matrix** $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ has eigenvalues $\lambda = 1$ and $\lambda = 0$.

Its eigenvectors are $\boldsymbol{x}_1 = (\mathbf{1}, \mathbf{1})$ and $\boldsymbol{x}_2 = (\mathbf{1}, -\mathbf{1})$. Then $P\boldsymbol{x}_1 = \boldsymbol{x}_1$ (steady state) and $P\boldsymbol{x}_2 = \mathbf{0}$ (nullspace). This example is a Markov matrix and a singular matrix (with $\lambda = 0$). Most important, P is a symmetric matrix. That makes its eigenvectors orthogonal.

1. **Markov matrix** : Each column of P adds to 1. Then $\lambda = 1$ is an eigenvalue.
2. $\boldsymbol{P}$ **is a singular matrix**. $\lambda = 0$ is an eigenvalue.
3. $\boldsymbol{P = P^{\mathrm{T}}}$ **is a symmetric matrix**. Perpendicular eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The only eigenvalues of a projection matrix are 0 and 1. The eigenvectors for $\lambda = 0$ (which means $P\boldsymbol{x} = 0\boldsymbol{x}$) fill up the nullspace. The eigenvectors for $\lambda = 1$ (which means $P\boldsymbol{x} = \boldsymbol{x}$) fill up the column space. The nullspace is projected to zero. The column space projects onto itself. *The projection keeps the column space and destroys the nullspace* :

$$\textbf{Project each part} \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \textbf{projects onto} \quad P\boldsymbol{v} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \end{bmatrix}$$

Projections have $\lambda = 0$ and 1. Permutations have all $|\lambda| = 1$. The next matrix E is a reflection and at the same time a permutation. E also has special eigenvalues.

Example 3 **The exchange matrix $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues 1 and -1.**

The eigenvector $(1,1)$ is unchanged by E. The second eigenvector is $(1,-1)$—its signs are reversed by E. A matrix with no negative entries can still have a negative eigenvalue! The eigenvectors for E are the same as for P, because $E = 2P - I$:

$$\boldsymbol{E} = 2\boldsymbol{P} - \boldsymbol{I} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4}$$

When a matrix is shifted by I, each λ is shifted by $\mathbf{1}$. No change in the eigenvectors.

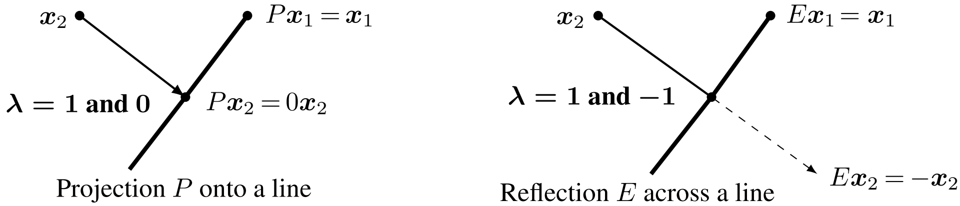


Figure 6.2: Projections P have eigenvalues 1 and 0. Exchanges E have $\lambda = 1$ and -1. A typical $\boldsymbol{x}$ changes direction, but an eigenvector $A\boldsymbol{x} = \lambda\boldsymbol{x}$ stays on the line through $\boldsymbol{x}$.

The Equation for the Eigenvalues: $\det(\boldsymbol{A} - \boldsymbol{\lambda I}) = 0$

For projection matrices we found λ's and $\boldsymbol{x}$'s by geometry: $P\boldsymbol{x} = \boldsymbol{x}$ and $P\boldsymbol{x} = \mathbf{0}$. For other matrices we use determinants and linear algebra. *This is the key calculation*—almost every application starts by solving $\det(A - \lambda I) = 0$ and $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

First move $\boldsymbol{\lambda x}$ to the left side. Write the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$ as $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$. The matrix $A - \lambda I$ times the eigenvector $\boldsymbol{x}$ is the zero vector. ***The eigenvectors make up the nullspace of $\boldsymbol{A - \lambda I}$***. When we know an eigenvalue λ, we find an eigenvector by solving $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$.

Eigenvalues first. If $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ has a nonzero solution, $A - \lambda I$ is not invertible. ***The determinant of $\boldsymbol{A - \lambda I}$ must be zero.*** This is how to recognize an eigenvalue λ:

Eigenvalues The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular.

Equation for the $\boldsymbol{n}$ eigenvalues of $\boldsymbol{A}$ $\quad \boxed{\det(\boldsymbol{A} - \boldsymbol{\lambda I}) = 0.}$ (5)

This "*characteristic polynomial*" $\det(A - \lambda I)$ involves only λ, not $\boldsymbol{x}$. Since λ appears all along the main diagonal of $A - \lambda I$, the determinant in (5) includes $(-\lambda)^n$. Then equation (5) has n solutions λ_1 to λ_n and ***A* has $\boldsymbol{n}$ eigenvalues**.

An n by n matrix has n eigenvalues (repeated $\boldsymbol{\lambda}$'s are possible !) Each λ leads to $\boldsymbol{x}$:

For each eigenvalue λ solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ or $A\boldsymbol{x} = \lambda\boldsymbol{x}$ to find an eigenvector $\boldsymbol{x}$.

Example 4 $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}$ is already singular (zero determinant). Find its λ's and $\boldsymbol{x}$'s.

When A is singular, $\lambda = 0$ is one of the eigenvalues. The equation $A\boldsymbol{x} = 0\boldsymbol{x}$ has solutions. They are the eigenvectors for $\lambda = 0$. But $\det(A - \lambda I) = 0$ is the way to find *all* λ's and $\boldsymbol{x}$'s. Always subtract λI from A:

$$\textit{Subtract } \boldsymbol{\lambda} \textit{ along the diagonal to find} \quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}. \tag{6}$$

Take the determinant "$ad - bc$" of this 2 by 2 matrix. From $1 - \lambda$ times $4 - \lambda$, the "ad" part is $\lambda^2 - 5\lambda + 4$. The "bc" part, not containing λ, is 2 times 2.

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(2) = \lambda^2 - 5\lambda. \tag{7}$$

Set this determinant $\boldsymbol{\lambda^2 - 5\lambda}$ to zero. One solution is $\lambda = 0$ (as expected, since A is singular). Factoring into λ times $\lambda - 5$, the other root is $\lambda = 5$:

$$\boxed{\det(A - \lambda I) = \lambda^2 - 5\lambda = 0 \quad \text{yields the eigenvalues} \quad \boldsymbol{\lambda_1 = 0} \quad \text{and} \quad \boldsymbol{\lambda_2 = 5}.}$$

Now find the eigenvectors. Solve $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ separately for $\lambda_1 = 0$ and $\lambda_2 = 5$:

$$(A - 0I)\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{-1} \end{bmatrix} \text{ for } \boldsymbol{\lambda_1} = \mathbf{0}$$

$$(A - 5I)\boldsymbol{x} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix} \text{ for } \boldsymbol{\lambda_2} = \mathbf{5}$$

The matrices $A - 0I$ and $A - 5I$ are singular (because 0 and 5 are eigenvalues). The eigenvectors $(2, -1)$ and $(1, 2)$ are in the nullspaces: $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ is $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

We need to emphasize: *There is nothing exceptional about* $\lambda = 0$. Like every other number, zero might be an eigenvalue and it might not. If A is singular, the eigenvectors for $\lambda = 0$ fill the nullspace: $A\boldsymbol{x} = 0\boldsymbol{x} = \mathbf{0}$. If A is invertible, zero is not an eigenvalue.

In the example, the shifted matrix $A - 5I$ is singular and 5 is the other eigenvalue.

Summary To solve the eigenvalue problem for an n by n matrix, follow these steps :

1. ***Compute the determinant of*** $A - \lambda I$. With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n.
2. ***Find the roots of this polynomial***, by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A. They make $A - \lambda I$ singular.
3. For each eigenvalue λ, ***solve*** $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ ***to find an eigenvector*** $\boldsymbol{x}$.

A note on the eigenvectors of 2 by 2 matrices. When $A - \lambda I$ is singular, both rows are multiples of a vector (a, b). *The eigenvector is any multiple of* $(b, -a)$. The example had

$\lambda = 0$: rows of $A - 0I$ in the direction $(1, 2)$; eigenvector in the direction $(2, -1)$

$\lambda = 5$: rows of $A - 5I$ in the direction $(-4, 2)$; eigenvector in the direction $(2, 4)$.

Previously we wrote that last eigenvector as $(1, 2)$. Both $(1, 2)$ and $(2, 4)$ are correct. There is a whole *line of eigenvectors*—any nonzero multiple of $\boldsymbol{x}$ is as good as $\boldsymbol{x}$. MATLAB's **eig**(A) divides by the length, to make each eigenvector $\boldsymbol{x}$ into a unit vector.

We must add a warning. Some 2 by 2 matrices have only *one* line of eigenvectors. This can only happen when two eigenvalues are equal. (On the other hand $A = I$ has equal eigenvalues and plenty of eigenvectors.) Without a full set of eigenvectors, we don't have a basis. We can't write every $\boldsymbol{v}$ as a combination of eigenvectors. In the language of the next section, *we can't diagonalize a matrix without n independent eigenvectors*.

Determinant and Trace

Bad news first: If you add a row of A to another row, or exchange rows, the eigenvalues usually change. *Elimination does not preserve the* λ's. The triangular U has *its* eigenvalues sitting along the diagonal—they are the pivots. But they are not the eigenvalues of A! Eigenvalues are changed when row 1 is added to row 2:

$$U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \quad \text{has } \lambda = 0 \text{ and } \lambda = 1; \quad A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{has } \lambda = 0 \text{ and } \lambda = 7.$$

Good news second: The *product* λ_1 *times* λ_2 *and the sum* $\lambda_1 + \lambda_2$ *can be found quickly from the matrix*. For this A, the product is 0 times 7. That agrees with the determinant of A (which is 0). The sum of eigenvalues is $0 + 7$. That agrees with the sum down the main diagonal (the **trace** is $1 + 6$). These quick checks always work.

> ***The product*** $(\boldsymbol{\lambda_1}) \cdots (\boldsymbol{\lambda_n})$ ***of the*** $\boldsymbol{n}$ ***eigenvalues*** *equals* ***the determinant of*** $\boldsymbol{A}$
> $\boldsymbol{\lambda_1 + \lambda_2 + \cdots + \lambda_n}$ *equals* ***the sum of the*** $\boldsymbol{n}$ ***diagonal entries = (trace of*** $\boldsymbol{A}$***)***

Those checks are very useful. They are proved in Problems 16–17 and again in the next section. They don't remove the pain of computing λ's. But when the computation is wrong, they generally tell us so. To compute the correct λ's, go back to $\det(A - \lambda I) = 0$.

The trace and determinant *do* tell everything when the matrix is 2 by 2. We never want to get those wrong! Here trace = $\mathbf{3}$ and $\det = \mathbf{2}$, so they all have $\lambda = \mathbf{1}$ and $\mathbf{2}$:

$$A = \begin{bmatrix} 1 & 9 \\ 0 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}. \tag{8}$$

That first A is one of the best matrices for finding eigenvalues: *because it is triangular*.

Why do the eigenvalues 1 and 2 of that triangular matrix lie along its diagonal?

Imaginary Eigenvalues

One more bit of news (not too terrible). The eigenvalues might not be real numbers.

Example 5 ***The 90° rotation*** $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ***has no real eigenvectors. Its eigenvalues are*** $\lambda_1 = i$ ***and*** $\lambda_2 = -i$. ***Then*** $\lambda_1 + \lambda_2 =$ **trace** $= 0$ ***and*** $\lambda_1\lambda_2 =$ ***determinant*** $= 1$.

After a rotation, *no real vector Qx stays in the same direction as x* ($x = 0$ is useless). There cannot be an eigenvector, unless we go to ***imaginary numbers***. Which we do.

To see how $i = \sqrt{-1}$ can help, look at Q^2 which is $-I$. If Q is rotation through 90°, then Q^2 is rotation through 180°. Its eigenvalues are -1 and -1. (Certainly $-Ix = -1x$.) Squaring Q will square each λ, so we must have $\boldsymbol{\lambda^2 = -1}$. *The eigenvalues of the* 90° *rotation matrix Q are $+i$ and $-i$*, because $i^2 = -1$.

Those λ's come as usual from $\det(Q - \lambda I) = 0$. This equation gives $\lambda^2 + 1 = 0$. Its roots are i and $-i$. We meet the imaginary numbers i and $-i$ also in the eigenvectors:

Complex eigenvectors $$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \boldsymbol{i} \end{bmatrix} = -\boldsymbol{i} \begin{bmatrix} \mathbf{1} \\ \boldsymbol{i} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{i} \\ \mathbf{1} \end{bmatrix} = \boldsymbol{i} \begin{bmatrix} \boldsymbol{i} \\ \mathbf{1} \end{bmatrix}.$$

Somehow these complex vectors $x_1 = (1, i)$ and $x_2 = (i, 1)$ keep their direction as they are rotated in complex space. Don't ask me how. This example makes the important point that real matrices can easily have complex eigenvalues and eigenvectors. The particular eigenvalues i and $-i$ also illustrate two properties of the special matrix Q.

1. Q is an orthogonal matrix so the absolute value of each λ is $|\lambda| = 1$.
2. Q is a skew-symmetric matrix so each λ is pure imaginary.

A symmetric matrix ($S^{\mathrm{T}} = S$) can be compared to a real number. A skew-symmetric matrix ($A^{\mathrm{T}} = -A$) can be compared to an imaginary number. An orthogonal matrix ($Q^{\mathrm{T}}Q = I$) corresponds to a complex number with $|\lambda| = 1$. For the eigenvalues of S and A and Q, those are more than analogies—they are facts to be proved.

The eigenvectors for all these special matrices are perpendicular. Somehow $(i, 1)$ and $(1, i)$ are perpendicular (Section 6.4 will explain the dot product of complex vectors).

Eigenvalues of AB and $A+B$

The first guess about the eigenvalues of AB is not true. An eigenvalue λ of A times an eigenvalue β of B usually does *not* give an eigenvalue of AB:

False proof $$ABx = A\beta x = \beta Ax = \beta\lambda x. \tag{9}$$

It seems that β times λ is an eigenvalue. When x is an eigenvector for A and B, this proof is correct. ***The mistake is to expect that A and B automatically share the same eigenvector x.*** Usually they don't. Eigenvectors of A are generally *not* eigenvectors of B.

A and B could have all zero eigenvalues while 1 is an eigenvalue of AB and $A + B$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad \text{then} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For the same reason, the eigenvalues of $A + B$ are generally not $\lambda + \beta$. Here $A + B$ has eigenvalues 1 and -1 while λ and β are zero. (At least the trace of $A + B$ is zero.)

The false proof suggests what is true. Suppose $\boldsymbol{x}$ really is an eigenvector for both A and B. Then we do have $AB\boldsymbol{x} = \lambda\beta\boldsymbol{x}$ and $BA\boldsymbol{x} = \lambda\beta\boldsymbol{x}$. When all n eigenvectors are shared by A and B, we *can* multiply eigenvalues and we find $AB = BA$. That test is important in quantum mechanics—time out to mention this application of linear algebra:

A and B share the same $\boldsymbol{n}$ *independent* eigenvectors if and only if $\boldsymbol{AB = BA}$.

Heisenberg's uncertainty principle In quantum mechanics, the position matrix P and the momentum matrix Q do not commute. In fact $QP - PQ = I$ (these are infinite matrices). To have $P\boldsymbol{x} = \boldsymbol{0}$ at the same time as $Q\boldsymbol{x} = \boldsymbol{0}$ would require $\boldsymbol{x} = I\boldsymbol{x} = \boldsymbol{0}$. If we knew the position exactly, we could not also know the momentum exactly. Problem 36 derives Heisenberg's uncertainty principle $\|P\boldsymbol{x}\|\,\|Q\boldsymbol{x}\| \geq \frac{1}{2}\|\boldsymbol{x}\|^2$.

■ REVIEW OF THE KEY IDEAS ■

1. $A\boldsymbol{x} = \lambda\boldsymbol{x}$ says that eigenvectors $\boldsymbol{x}$ keep the same direction when multiplied by A.
2. $A\boldsymbol{x} = \lambda\boldsymbol{x}$ also says that $\det(A - \lambda I) = 0$. This equation determines n eigenvalues.
3. The sum of the λ's equals the sum down the main diagonal of A (**the trace**). The product of the λ's equals the determinant of A.
4. Projections P, exchanges E, $90°$ rotations Q have eigenvalues $1, 0, -1, i, -i$. Singular matrices have $\lambda = 0$. Triangular matrices have λ's on their diagonal.
5. *Special properties of a matrix lead to special eigenvalues and eigenvectors.* That is a major theme of Chapter 6.

■ WORKED EXAMPLES ■

6.1 A Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A + 4I$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace $\lambda_1 + \lambda_2 = 4$ and the determinant λ_1 times $\lambda_2 = 3$.

Solution The eigenvalues of A come from $\det(A - \lambda I) = 0$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} \mathbf{2-\boldsymbol{\lambda}} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{2-\boldsymbol{\lambda}} \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into $(\lambda - 1)(\lambda - 3) = 0$ so the eigenvalues of A are $\lambda_1 = \mathbf{1}$ and $\lambda_2 = \mathbf{3}$. For the trace, the sum $2+2$ agrees with $1+3$. The determinant 3 agrees with the product $\lambda_1\lambda_2$.

The eigenvectors come separately by solving $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$ which is $A\boldsymbol{x} = \lambda\boldsymbol{x}$:

$$\boldsymbol{\lambda = 1:} \quad (A - I)\boldsymbol{x} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{\lambda = 3:} \quad (A - 3I)\boldsymbol{x} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A^2 and A^{-1} and $A + 4I$ keep the *same eigenvectors as A itself*. Their eigenvalues are λ^2 and λ^{-1} and $\lambda + 4$:

$$A^2 \text{ has eigenvalues } 1^2 = 1 \text{ and } 3^2 = 9 \qquad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \qquad A + 4I \text{ has } \begin{matrix} 1 + 4 = 5 \\ 3 + 4 = 7 \end{matrix}$$

Notes for later sections: A has *orthogonal eigenvectors* (Section 6.3 on symmetric matrices). A can be *diagonalized* since $\lambda_1 \neq \lambda_2$ (Section 6.2). A is *similar* to every 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.2). A is a *positive definite matrix* (Section 6.3) since $A = A^{\mathrm{T}}$ and the λ's are positive.

6.1 B **How can you estimate the eigenvalues of any $\boldsymbol{A}$?** Gershgorin gave this answer.

Every eigenvalue of A must be "near" at least one of the entries a_{ii} on the main diagonal. For λ to be "near a_{ii}" means that $|a_{ii} - \lambda|$ is no more than **the sum $\boldsymbol{R_i}$ of all other $|\boldsymbol{a_{ij}}|$ in that row $\boldsymbol{i}$ of the matrix**. Then R_i is the radius of a circle centered at a_{ii}.

Every $\boldsymbol{\lambda}$ is in the circle around one or more diagonal entries $\boldsymbol{a_{ii}}$: $|\boldsymbol{a_{ii}} - \boldsymbol{\lambda}| \leq \boldsymbol{R_i}$.

Here is the reasoning. When λ is an eigenvalue, $A - \lambda I$ is not invertible. Then $A - \lambda I$ cannot be diagonally dominant (see Section 2.5). So at least one diagonal entry $a_{ii} - \lambda$ is *not larger* than the sum R_i of all other entries $|a_{ij}|$ (we take absolute values!) in row i.

Example Every eigenvalue λ of this A falls into one or both of the **Gershgorin circles**: The centers are a and d. The radii of the circles are $R_1 = |b|$ and $R_2 = |c|$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{matrix} \text{First circle:} & |\lambda - a| \leq |b| \\ \text{Second circle:} & |\lambda - d| \leq |c| \end{matrix}$$

Those are circles in the complex plane, since λ could certainly be a complex number.

6.1 C Find the eigenvalues and eigenvectors of this symmetric 3 by 3 matrix S:

Symmetric matrix
Singular matrix
Trace $\mathbf{1+2+1=4}$

$$S = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solution Since all rows of S add to zero, the vector $\boldsymbol{x} = (1,1,1)$ gives $S\boldsymbol{x} = \mathbf{0}$. This is an eigenvector for $\lambda = 0$. To find λ_2 and λ_3, compute the 3 by 3 determinant:

$$\det(S - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \begin{aligned} &= (1-\lambda)(2-\lambda)(1-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)[(2-\lambda)(1-\lambda) - 2] \\ &= \boldsymbol{(1-\lambda)(-\lambda)(3-\lambda)}. \end{aligned}$$

Those three factors give $\lambda = 0, 1, 3$ as the solutions to $\det(S - \lambda I) = 0$. Each eigenvalue corresponds to an eigenvector (or a line of eigenvectors):

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad S\boldsymbol{x}_1 = \mathbf{0}\boldsymbol{x}_1 \qquad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad S\boldsymbol{x}_2 = \mathbf{1}\boldsymbol{x}_2 \qquad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad S\boldsymbol{x}_3 = \mathbf{3}\boldsymbol{x}_3 \,.$$

I notice again that eigenvectors are perpendicular when S is symmetric. We were lucky to find $\lambda = 0, 1, 3$. For a larger matrix I would use **eig**(A), and never touch determinants.

The full command $[\boldsymbol{X},\boldsymbol{E}] = \textbf{eig}(A)$ will produce unit eigenvectors in the columns of X.

Problem Set 6.1

1 The example at the start of the chapter has powers of this matrix A:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Find the eigenvalues of these matrices. All powers have the same eigenvectors. Show from A how a row exchange can produce different eigenvalues.

2 Find the eigenvalues and eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

$A + I$ has the _____ eigenvectors as A. Its eigenvalues are _____ by 1.

3 Compute the eigenvalues and eigenvectors of A and A^{-1}. Check the trace !

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the _____ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues _____.

4 Compute the eigenvalues and eigenvectors of A and A^2:

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same _____ as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues _____. In this example, how do you see that $\lambda_1^2 + \lambda_2^2 = 13$?

5 Find the eigenvalues of A and B (easy for triangular matrices) and $A + B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Eigenvalues of $A + B$ (*are equal to*)(*are not equal to*) eigenvalues of A plus eigenvalues of B.

6 Find the eigenvalues of A and B and AB and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

(a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

(b) Are the eigenvalues of AB equal to the eigenvalues of BA?

7 Elimination produces $A = LU$. The eigenvalues of U are on its diagonal; they are the _____. The eigenvalues of L are on its diagonal; they are all _____. The eigenvalues of A are not the same as _____.

8 (a) If you know that $\boldsymbol{x}$ is an eigenvector, the way to find λ is to _____.

(b) If you know that λ is an eigenvalue, the way to find $\boldsymbol{x}$ is to _____.

9 What do you do to the equation $A\boldsymbol{x} = \lambda\boldsymbol{x}$, in order to prove (a), (b), and (c)?

(a) λ^2 is an eigenvalue of A^2, as in Problem 4.

(b) λ^{-1} is an eigenvalue of A^{-1}, as in Problem 3.

(c) $\lambda + 1$ is an eigenvalue of $A + I$, as in Problem 2.

10 Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞. Explain from those answers why A^{100} is close to A^∞:

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix} = \text{limit of } A^n$$

11 Here is a strange fact about 2 by 2 matrices with eigenvalues $\lambda_1 \neq \lambda_2$: The columns of $A - \lambda_1 I$ are multiples of the eigenvector $\boldsymbol{x}_2$. Any idea why this should be?

12 Find three eigenvectors for this matrix P (projection matrices have $\lambda=1$ and 0):

$$\textbf{Projection matrix} \qquad P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ, so do all their linear combinations. Find an eigenvector of P with no zero components.

13 From the unit vector $\boldsymbol{u} = \left(\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$ construct the rank one projection matrix $P = \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$. This matrix has $P^2 = P$ because $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$.

(a) $P\boldsymbol{u}=\boldsymbol{u}$ comes from $(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u}=\boldsymbol{u}(\_\_\_\_\_)$. Then $\lambda = 1$.

(b) If $\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ show that $P\boldsymbol{v} = \mathbf{0}$. Then $\lambda = 0$.

(c) Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.

14 Solve $\det(Q - \lambda I) = 0$ by the quadratic formula to reach $\lambda = \cos\theta \pm i \sin\theta$:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{rotates the } xy \text{ plane by the angle } \theta. \text{ No real } \lambda\text{'s.}$$

Find the eigenvectors of Q by solving $(Q - \lambda I)\boldsymbol{x} = \mathbf{0}$. Use $i^2 = -1$.

15 Every permutation matrix leaves $\boldsymbol{x} = (1, 1, \ldots, 1)$ unchanged. Then $\lambda = 1$. Find two more λ's (possibly complex) for these permutations, from $\det(P - \lambda I) = 0$:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \qquad \textbf{All } |\boldsymbol{\lambda}| = \mathbf{1}$$

16 **The determinant of A equals the product** $\lambda_1\lambda_2\cdots\lambda_n$. Start with the polynomial $\det(A - \lambda I)$ separated into its n factors $\lambda_2 - \lambda$ (always possible). Then set $\lambda = 0$:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) \quad \text{so} \quad \det A = \_\_\_\_\_.$$

Check this rule for $\det A$ in Example 1 where the Markov matrix has $\lambda = 1$ and $\frac{1}{2}$.

17 The sum of the diagonal entries (the **trace**) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \lambda^2 - (a + d)\lambda + ad - bc = 0.$$

The quadratic formula gives the eigenvalues $\lambda = (a+d+\sqrt{})/2$ and $\lambda = \_\_\_\_\_$. Their sum is _____. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = \_\_\_\_\_$.

18 If A has $\lambda_1 = 4$ and $\lambda_2 = 5$ then $\det(A - \lambda I) = (\lambda - 4)(\lambda - 5) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $a + d = 9$ and determinant 20 and $\lambda = 4, 5$.

19 A 3 by 3 matrix B is known to have eigenvalues $0, 1, 2$. This information is enough to find three of these (give the answers where possible) :

(a) the rank of B
(b) the determinant of $B^{\mathrm{T}}B$
(c) the eigenvalues of $B^{\mathrm{T}}B$
(d) the eigenvalues of $(B^2 + I)^{-1}$.

20 Choose the last rows of A and C to give eigenvalues $4, 7$ and $1, 2, 3$:

Companion matrices $$A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}.$$

21 ***The eigenvalues of A equal the eigenvalues of*** $\boldsymbol{A^{\mathrm{T}}}$. This is because $\det(A - \lambda I)$ equals $\det(A^{\mathrm{T}} - \lambda I)$. That is true because _____. Show by an example that the eigenvectors of A and A^{T} are *not* the same.

22 Construct any 3 by 3 Markov matrix M: positive entries down each column add to 1. Show that $M^{\mathrm{T}}(1, 1, 1) = (1, 1, 1)$. By Problem 21, $\lambda = 1$ is also an eigenvalue of M. Challenge: A 3 by 3 singular Markov matrix with trace $\frac{1}{2}$ has what λ's ?

23 Find three 2 by 2 matrices that have $\lambda_1 = \lambda_2 = 0$. The trace is zero and the determinant is zero. A might not be the zero matrix but check that $A^2 = 0$.

24 This matrix is singular with rank one. Find three λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

25 Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. **Then** $\boldsymbol{A = B}$. *Reason*: Any vector $\boldsymbol{x}$ is a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. What is $A\boldsymbol{x}$? What is $B\boldsymbol{x}$?

26 The block B has eigenvalues $1, 2$ and C has eigenvalues $3, 4$ and D has eigenvalues $5, 7$. Find the eigenvalues of the 4 by 4 matrix A:

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

27 Find the rank and the four eigenvalues of A and C:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

28 Subtract I from A in Problem 27. Find the λ's and then the determinants of

$$B = A - I = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = I - A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

29 (Review) Find the eigenvalues of A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

30 When $a + b = c + d$ show that $(1, 1)$ is an eigenvector and find both eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

31 If we exchange rows 1 and 2 *and* columns 1 and 2, the eigenvalues don't change. Find eigenvectors of A and B for $\lambda = 11$. *Rank one gives* $\lambda_2 = \lambda_3 = 0$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \quad \text{and} \quad B = PAP^{\mathrm{T}} = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}.$$

32 Suppose A has eigenvalues $0, 3, 5$ with independent eigenvectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Find all solutions.

(c) $A\boldsymbol{x} = \boldsymbol{u}$ has no solution. If it did then _____ would be in the column space.

Challenge Problems

33 Show that $\boldsymbol{u}$ is an eigenvector of the rank one 2×2 matrix $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. Find both eigenvalues of A. Check that $\lambda_1 + \lambda_2$ agrees with the trace $u_1 v_1 + u_2 v_2$.

34 Find the eigenvalues of this permutation matrix P from $\det(P - \lambda I) = 0$. Which vectors are not changed by the permutation? They are eigenvectors for $\lambda = 1$. Can you find three more eigenvectors?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

35 There are six 3 by 3 permutation matrices P. What numbers can be the *determinants* of P? What numbers can be *pivots*? What numbers can be the *trace* of P? What *four numbers* can be eigenvalues of P, as in Problem 15?

36 (**Heisenberg's Uncertainty Principle**) $AB - BA = I$ can happen for infinite matrices with $A = A^{\mathrm{T}}$ and $B = -B^{\mathrm{T}}$. Then

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}AB\boldsymbol{x} - \boldsymbol{x}^{\mathrm{T}}BA\boldsymbol{x} \le 2\|A\boldsymbol{x}\|\,\|B\boldsymbol{x}\|.$$

Explain that last step by using the Schwarz inequality $|\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}| \le ||\boldsymbol{u}||\,||\boldsymbol{v}||$. Then Heisenberg's inequality says that $\|A\boldsymbol{x}\|/\|\boldsymbol{x}\|$ *times* $\|B\boldsymbol{x}\|/\|\boldsymbol{x}\|$ is at least $\frac{1}{2}$. It is impossible to get the position error and momentum error both very small.

37 *Find a* 2 *by* 2 *rotation matrix* (other than I) *with* $A^3 = I$. Its eigenvalues must satisfy $\lambda^3 = 1$. They can be $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. What are the trace and determinant of A?

38 (a) Find the eigenvalues and eigenvectors of A. They depend on c:

$$A = \begin{bmatrix} .4 & 1-c \\ .6 & c \end{bmatrix}.$$

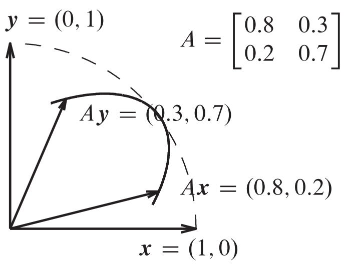
(b) Show that A has just one line of eigenvectors when $c = 1.6$.

(c) This is a Markov matrix when $c = 0.8$. Then A^n approaches what matrix A^∞?

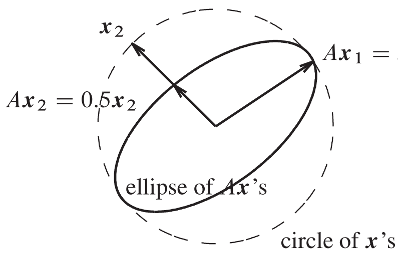
Eigshow in MATLAB

There is a MATLABdemo (just type **eigshow**), displaying the eigenvalue problem for a 2 by 2 matrix. It starts with the unit vector $\boldsymbol{x} = (1, 0)$. *The mouse makes this vector move around the unit circle*. At the same time the screen shows $A\boldsymbol{x}$, in color and also moving. Possibly $A\boldsymbol{x}$ is ahead of $\boldsymbol{x}$. Possibly $A\boldsymbol{x}$ is behind $\boldsymbol{x}$. *Sometimes $A\boldsymbol{x}$ is parallel to $\boldsymbol{x}$*.

At that parallel moment, $A\boldsymbol{x} = \lambda\boldsymbol{x}$ (at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in the second figure).



These are not eigenvectors



$A\boldsymbol{x}$ lines up with $\boldsymbol{x}$ at eigenvectors

The eigenvalue λ is the length of $A\boldsymbol{x}$, when the unit eigenvector $\boldsymbol{x}$ lines up. The built-in choices for A illustrate three possibilities: $0, 1$, or 2 real vectors where $A\boldsymbol{x}$ crosses $\boldsymbol{x}$.

The axes of the ellipse are **singular vectors** in Section 7.1.